On Nikodým, Grothendieck, Vitali-Hahn-Saks and Valdivia Measure Theorems

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Let $\mathcal{A}$ be an algebra of subsets of a set Ω and let $ba(\mathcal{A})$ the Banach space of real or complex bounded finitely additive measures defined on $\mathcal{A}$ endowed with the variation norm, which is equivalent to the supremum norm.

A subset $\mathcal{B}$ of $\mathcal{A}$ has property (N) if each $\mathcal{B}$-poinwise bounded sequence $(\mu_n, n \in \mathbb{N})$ of $ba(\mathcal{A})$ is norm bounded in $ba(\mathcal{A})$, i.e., the sequence $(\mu_n, n \in \mathbb{N})$ is uniformly bounded in $\mathcal{A}$. $\mathcal{B}$ has property (G) $[(VHS)]$ if for each bounded sequence [if for each sequence] $(\mu_n, n \in \mathbb{N})$ of $ba(\mathcal{A})$ the $\mathcal{B}$-poinwise convergence of $(\mu_n, n \in \mathbb{N})$ to $\mu \in ba(\mathcal{A})$ implies its weak convergence, i.e., $\lim_{n\to\infty} \varphi(\mu_n) = \varphi(\mu)$ for each linear continuous form φ defined in $ba(\mathcal{A})$.

$\mathcal{B}$ has property (sN) $[(sG)$ or $(sVHS)]$ if every increasing covering $\{\mathcal{B}_n : n \in \mathbb{N}\}$ of $\mathcal{B}$ contains a set $\mathcal{B}_p$ with property (N) $[(G)$ or $(VHS)]$, and $\mathcal{B}$ has property (wN) $[(wG)$ or $(wVHS)]$ if for every increasing web $\{\mathcal{B}_{n_1 n_2 \cdots n_m} : n_i \in \mathbb{N}, 1 \leq i \leq m, m \in \mathbb{N}\}$ of $\mathcal{B}$ there exists sequence $(p_n, n \in \mathbb{N})$ of natural numbers such that for each natural number m the set $\mathcal{B}_{p_1 p_2 \cdots p_m}$ property (N) $[(G)$ or $(VHS)]$ for every $m \in \mathbb{N}$.

The classical theorems of Nikodým-Grothendieck, Valdivia, Grothendieck and Vitali-Hahn-Saks say, respectively, that every σ-algebra $\mathcal{A}$ has properties (N), (sN), (G) and (VHS). The main objective of this conference is to expose our recent results that every σ-algebra $\mathcal{A}$ has properties (wN), (wG) and $(wVHS)$, improving the mentioned four theorems. Moreover applications of these new properties are considered as well as the characterization that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $(wWHS)$ if and only if $\mathcal{B}$ has property (wN) and $\mathcal{A}$ has property (G) and several open questions.

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